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Strongly coupled 't Hooft model on the lattice *

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A lattice strong coupling calculation of the spectrum and chiral condensate of the 't Hooft model is presented. The agreement with the results of the continuum theory is strikingly good even at the fourth order in the strong coupling expansions.

1. Introduction

Several attempts have been made to analyze the spectrum of two-dimensional quantum chromodynamics (QCD_2). 't Hooft [1] was able to find the meson spectrum for $U(\mathcal{N}_c)$ QCD_2 using the $1/\mathcal{N}_c$ expansion. The results, obtained in the weak coupling limit with $g^2\mathcal{N}_c$ fixed and $\mathcal{N}_c$ approaching infinity, revealed that there is a spectrum of color singlet mesonic states with equal energy spacing. Later, Witten [2] explained how to fit baryons into this picture, showing that they can be interpreted as the 't Hooft-Polyakov monopoles of the theory.

In this contribution we report on a lattice computation on the spectrum and chiral condensate of the 't Hooft model in the strong coupling limit. The strong coupling limit of gauge theories is highly nonuniversal; in spite of this difficulty there exist strong coupling computations which claim some degree of success [3]. We compute analytically the mass of the scalar and the pseudoscalar mesons and the chiral condensate up to the fourth order in a strong coupling expansion. We find also that Witten's interpretation for the baryons is consistent with our lattice results.

The continuum one-flavor 't Hooft model is defined by the Euclidean action

$$S=\int\left[\overline{\psi}_a\gamma_\mu(\partial_\mu\psi_a+A^b_{a\mu}\psi_b)-\frac{1}{4g^2}F^b_{\mu\nu a}F^a_{\mu\nu b}\right]d^2x \tag{1}$$

where $a,b=1,\dots\mathcal{N}_c$.

The Hamiltonian and the Gauss constraints are

$$H=\int\left[\frac{g^2}{2}(E^A(x))^2+\overline{\psi}_a\alpha(i\partial_x\psi_a+A^b_{a\ x}\psi_b)\right]dx \tag{2}$$

$$\partial_xE^A(x)+ig[A^A(x),E^A(x)]+\psi^\dagger_aT^A_{ab}\psi_b(x)\sim 0 \tag{3}$$

with $A=0,\dots,\mathcal{N}_c^2-1$ and the electric field operators $E^A(x)$ satisfying the group algebra. The lattice Hamiltonian and Gauss constraints reproducing Eqs.(2, 3) in the naive continuum limit, read

$$H=\frac{g^2a}{2}\sum_{x=1}^{N}E^A_xE^A_x-\frac{it}{2a}[R-L]\quad , \tag{4}$$

$$E^A_x-U^\dagger(x-1)E^A_{x-1}U(x-1)+ + \sum_{a,b=1}^{\mathcal{N}_c}\psi^\dagger_aT^A_{ab}\psi_b(x)-\frac{\mathcal{N}_c}{2}\delta^{A,0}\sim 0 \tag{5}$$

where the right and left hopping operators are defined ($L=R^\dagger$) by

$$R=\sum_{x=1}^{N}R_x=\sum_{x=1}^{N}\sum_{a,b=1}^{\mathcal{N}_c}\psi^\dagger_{a,x+1}U_{ab}(x)\psi_{b,x} \tag{6}$$

and the matrix $U(x)$, associated with the link $[x,x+1]$, is a group element of $U(\mathcal{N}_c)$ in the fundamental representation.

The Hamiltonian (4), rescaled by the factor $g^2a/2$, can be written as

$$H=H_0+\epsilon H_h \tag{7}$$

with $H_0=\sum_{x=1}^{N}E^A_xE^A_x$, $H_h=-i(R-L)$ and $\epsilon=t/g^2a^2$ the expansion parameter. Since H_0 and H_h are both gauge invariant, if one finds a

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gauge invariant eigenstate of H_0, perturbations in H_h retain gauge invariance. Due to the Gauss law constraints (5) the lowest energy eigenstate of H_0 is a color singlet with a density of states of $\mathcal{N}_c/2$ fermions per site. A color singlet at a given site can be formed either by leaving it unoccupied or by putting on it $\mathcal{N}_c$ fermions by means of the creation operator $S_+(x) = \epsilon_{a_1 \dots a_{\mathcal{N}_c}} \psi^\dagger_{a_1 x} \cdots \psi^\dagger_{a_{\mathcal{N}_c}}$. The $N/2$ singlets can be distributed arbitrarily among the N sites so that there are $N!/(N/2)!$ degenerate ground states. The local fermion number operator

$$\rho_x = \sum_{a=1}^{\mathcal{N}_c} \psi^\dagger_{ax}\psi_{ax} - \mathcal{N}_c/2 \tag{8}$$

takes the value $+\mathcal{N}_c/2$ on occupied sites and $-\mathcal{N}_c/2$ on empty sites.

2. Hadron spectrum

First order perturbations to the vacuum energy vanish. The ground state degeneracy is removed at the second order in the strong coupling expansion. The vacuum energy - at order ϵ^2 - reads

$$E_0^{(2)} = \langle H_h^\dagger \frac{\Pi}{E_0^{(0)} - H_0} H_h \rangle \quad , \tag{9}$$

where the expectation values are defined on the degenerate subspace and Π is the operator projecting on a set orthogonal to the degenerate ground states. The commutator $[H_0, H_h] = C_2^f(\mathcal{N}_c) H_h$, where $C_2^f(\mathcal{N}_c) = \mathcal{N}_c/2$ is the quadratic Casimir of the fundamental representation of $U(\mathcal{N}_c)$, holds on the degenerate subspace. Using the commutator between H_0 and H_h from Eq.(9), one gets

$$E_0^{(2)} = -\frac{2}{C_2^f(\mathcal{N}_c)} \langle RL \rangle \ . \tag{10}$$

The vacuum expectation value $\langle\ ,\ \rangle$ is the inner product in the full Hilbert space and is defined as $\langle\ ,\ \rangle = \prod_x \int dU_x(\ ,\ )$, where dU is the Haar measure on the gauge group manifold and $(\ ,\ )$ is the fermion Fock space inner product. Performing the integrals over the group elements in Eq.(10) the combination RL can be written as a spin-$\mathcal{N}_c/2$ Ising Hamiltonian: in fact, taking into account that products of L_x and R_y at different points have vanishing expectation values, one can rewrite Eq.(10) as

$$E_0^{(2)} = \frac{4}{\mathcal{N}_c^2} \ \langle \sum_{x=1}^{N} \rho(x)\rho(x+1) - \frac{1}{4}\mathcal{N}_c^2 N \ \rangle \quad . \tag{11}$$

The Hamiltonian in Eq.(11) is an antiferromagnetic Ising Hamiltonian in the space of pure fermion states where $\rho_x = \pm\mathcal{N}_c/2$ and $\sum_{x=1}^N \rho_x = 0$. The Hamiltonian in Eq.(11) has two degenerate ground states characterized by a fermion distribution $\rho_x = \pm\mathcal{N}_c/2(-1)^x$.

Let us now investigate the one-flavor 't Hooft model meson spectrum in the strong coupling limit. We evaluate the ground state energy up to the fourth order in the strong coupling expansion

$$\begin{aligned} E_{g.s.} &= \frac{g^2 a}{2}(E^{(0)}_{g.s.} + \epsilon^2 E^{(2)}_{g.s.} + \epsilon^4 E^{(4)}_{g.s.}) \\ &= \frac{g^2 a}{2}(-2\mathcal{N}_c^2 N\epsilon^2 + 16\mathcal{N}_c^2 N\epsilon^4) \ . \end{aligned} \tag{12}$$

The lowest lying excitations are a pseudoscalar and a scalar created by the Fourier transform of the conserved gauge invariant currents at zero momentum $\sum_x j_1(x) = R + L$ and $\sum_x j_5(x) = R - L$, respectively.

$$|P\rangle = (R+L)|g.s.\rangle \ , \quad |S\rangle = (R-L)|g.s.\rangle \tag{13}$$

At the zero-th order they are degenerate, but the degeneracy is removed at the second order in the strong coupling expansion. The mass of these low lying excitations can be obtained by computing their energies and by subtracting the ground state energy (12).

Up to the fourth order in ϵ the mass of the state $|P\rangle$ is given by

$$m_P = \frac{g^2 a}{2}(\frac{1}{2} - 4\epsilon^2 + 80\epsilon^4) \tag{14}$$

whereas the one of the scalar meson $|S\rangle$ is

$$m_S = \frac{g^2 a}{2}(\frac{1}{2} + 4\epsilon^2 + 80\epsilon^4). \tag{15}$$

In Eqs.(14,15) the coupling constant has been rescaled according to $g^2 \longrightarrow g^2\mathcal{N}_c$ so that the strong coupling expansion parameter changes as

$\epsilon \longrightarrow \frac{\epsilon}{\mathcal{N}_c}$. This rescaling is needed, since the meson masses are proportional to $\mathcal{N}_c$ at each order in the strong coupling expansion. This infinity can be absorbed in the definition of the coupling constant to produce a smooth large-$\mathcal{N}_c$ limit.

The spectrum exhibits also one baryon which can be created at zero momentum by acting on the ground state with the color singlet operator $B^\dagger$

$$|B>= B^\dagger|g.s.>= \sum_{x=1}^{N} \psi^\dagger_{1x}\psi^\dagger_{2x}\ldots\psi^\dagger_{\mathcal{N}_c x}|g.s.> \quad .(16)$$

At the zero-th order in the strong coupling expansion the baryon is massless, since the creation operator $B^\dagger$ does not contain any color flux ($H_0|B>=0$). At the second order the baryon acquires a mass $m_B^{(2)} = g^2 a$. This result is in agreement with Witten's conjecture [2] that the baryons are the solitons of the theory: baryons have a mass proportional to the inverse of the coupling constant (in strong coupling at the zero-th order they are massless) but acquire a mass proportional to g^2 already at the second order.

3. Chiral symmetry breaking

In the continuum 't Hooft model, the chiral symmetry is dynamically broken by the anomaly. The order parameter is the mass operator $M(x) = \overline{\psi}_a(x)\psi_a(x)$, which acquires a nonzero vacuum expectation value, giving rise to the chiral condensate which in the large-$\mathcal{N}_c$ limit reads [4]

$$\chi_c = \langle\overline{\psi}\psi\rangle = -\mathcal{N}_c(\frac{g_c^2\mathcal{N}_c}{12\pi})^{\frac{1}{2}} \quad . \tag{17}$$

In this section we shall exhibit the result of the computation of the lattice chiral condensate χ_L up to the fourth order in the strong coupling expansion. In the staggered fermion formalism χ_L is given by the expectation value of the mass operator

$$M = -\frac{1}{Na}\sum_{x=1}^{N}\sum_{a=1}^{\mathcal{N}_c}(-1)^x\psi^\dagger_{ax}\psi_{ax} \tag{18}$$

on the perturbed states. To the fourth order in ϵ, χ_L is given by

$$\chi_L = -\frac{1}{a}\mathcal{N}_c(\frac{1}{2} - 8\epsilon^2 + 32\epsilon^4) \ . \tag{19}$$

4. Concluding Remarks

We shall now compare the strong coupling results with the continuum theory. We have to extrapolate the strong coupling series, derived under the assumption that the parameter $\epsilon^2 = t^2/g^4a^4 \ll 1$, to the region in which $\epsilon^2 \gg 1$; this region corresponds to the continuum theory since when $a \longrightarrow 0$, $\epsilon^2 \longrightarrow \infty$ for a given g. For this purpose it is customary to make use of Padé approximants.

We first compute the lattice light velocity by equating the lattice chiral condensate Eq.(19) to its continuum counterpart Eq.(17); we get $t = 0.9757$ which lies 2.4% below the exact answer $t = 1$. Applying the same procedure to the mass of the pseudoscalar excitation the extrapolated value of m_P is $m_P/g = 0.6655$ which agrees within 16% with the result obtained in [5] in the continuum for $\mathcal{N}_c = 3$ and with the lattice numerical calculations of Ref. [6].

The strong coupling perturbation expansion even for the 't Hooft model provides, not only accurate numerical results for the spectrum and chiral condensate, but also leads to a simple and correct understanding of confinement and to an intuitive picture of the vacuum of gauge theories.

REFERENCES

1. G. 't Hooft, Nucl. Phys. **B75**, 461 (1974).
2. E. Witten, Nucl. Phys. **B160**, 57 (1979).
3. F. Berruto, G. Grignani, G. W. Semenoff and P. Sodano, Phys. Rev. **D57**, 5070 (1998); Phys. Rev. **D59**, 034504 (1999); Annals of Phys. **275**, 254 (1999) and references therein.
4. A. R. Zhitnisky, Phys. Lett. **B165**, 405 (1985).
5. G. Bhattacharya, Nucl. Phys. **B205**, 461 (1982).
6. J-Q. Jiang, X-Q. Luo, Z-H. Mei, H. Jirari, H. Kröger and C-M. Wu, Phys. Rev. **D60**, 014501 (1999).